

Degree sum conditions for graphs to have proper connection number 2*

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Abstract

A path P in an edge-colored graph G is a *proper path* if no two adjacent edges of P are colored with the same color. The graph G is *proper connected* if, between every pair of vertices, there exists a proper path in G . The *proper connection number* $pc(G)$ of a connected graph G is defined as the minimum number of colors to make G proper connected. In this paper, we study the degree sum condition for a general graph or a bipartite graph to have proper connection number 2. First, we show that if G is a connected noncomplete graph of order $n \geq 5$ such that $d(x) + d(y) \geq \frac{n}{2}$ for every pair of nonadjacent vertices $x, y \in V(G)$, then $pc(G) = 2$ except for three small graphs on 6, 7 and 8 vertices. In addition, we obtain that if G is a connected bipartite graph of order $n \geq 4$ such that $d(x) + d(y) \geq \frac{n+6}{4}$ for every pair of nonadjacent vertices $x, y \in V(G)$, then $pc(G) = 2$. Examples are given to show that the above conditions are best possible.

Keywords: proper connection number; proper connection coloring; bridge-block tree; degree sum condition.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [2] for graph theoretical notation and terminology not described here. Let G be a graph.

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We use $V(G)$, $E(G)$, $|G|$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, number of vertices, maximum degree and minimum degree of G , respectively. For any two disjoint subsets X and Y of $V(G)$, we denote by $E_G(X, Y)$ the set of edges of G that have one end in X and the other in Y , and denote by $|E_G(X, Y)|$ the number of edges in $E_G(X, Y)$. For $v \in V(G)$, let $N(v)$ denote the set of neighbours, and $d_H(v)$ denote the degree of v in subgraph H of G .

Let G be a nontrivial connected graph with an associated *edge-coloring* $c : E(G) \rightarrow \{1, 2, \dots, t\}$, $t \in \mathbb{N}$, where adjacent edges may have the same color. If adjacent edges of G are assigned different colors by c , then c is a *proper (edge-)coloring*. For a graph G , the minimum number of colors needed in a proper coloring of G is referred to as the *edge-chromatic number* of G and denoted by $\chi'(G)$. A path of an edge-colored graph G is said to be a *rainbow path* if no two edges on the path have the same color. The graph G is called *rainbow connected* if every pair of distinct vertices of G is connected by a rainbow path in G . An edge-coloring of a connected graph is a *rainbow connection coloring* if it makes the graph rainbow connected. This concept of rainbow connection of graphs was introduced by Chartrand et al. [4] in 2008. The *rainbow connection number* $rc(G)$ of a connected graph G is the smallest number of colors that are needed in order to make G rainbow connected. The readers who are interested in this topic can see [9, 10] for a survey.

Inspired by rainbow connection coloring and proper coloring in graphs, Andrews et al. [1] and Borozan et al. [3] introduced the concept of proper-path coloring. Let G be a nontrivial connected graph with an edge-coloring. A path in G is called a *proper path* if no two adjacent edges of the path receive the same color. An edge-coloring c of a connected graph G is a *proper connection coloring* if every pair of distinct vertices of G are connected by a proper path in G . And if k colors are used, then c is called a *proper connection k -coloring*. An edge-colored graph G is *proper connected* if any two vertices of G are connected by a proper path. For a connected graph G , the minimum number of colors that are needed in order to make G proper connected is called the *proper connection number* of G , denoted by $pc(G)$. Let G be a nontrivial connected graph of order n and size m , then we have that $1 \leq pc(G) \leq \min\{\chi'(G), rc(G)\} \leq m$. Furthermore, $pc(G) = 1$ if and only if $G = K_n$ and $pc(G) = m$ if and only if $G = K_{1,m}$ as a star of size m . For more details, we refer to [5, 7, 11] and a dynamic survey [8].

In [1], the authors considered many conditions on G which force $pc(G)$ to be small, in particular $pc(G) = 2$. Recently, Huang et al. presented minimum degree condition for a graph to have proper connection number 2 in [6]. They showed that if G is a connected noncomplete graph of order $n \geq 5$ with $\delta(G) \geq n/4$, then $pc(G) = 2$ except for two small graphs on 7 and 8 vertices. In addition, they obtained that if G

is a connected bipartite graph of order $n \geq 4$ with $\delta(G) \geq \frac{n+6}{8}$, then $pc(G) = 2$. It is worth mentioning that the two bounds on the minimum degree in the above results are best possible. On the other hand, in [12], the authors showed that if G is a graph with n vertices such that $\delta(G) \geq \frac{n-1}{2}$, then G has a Hamiltonian path. It is also known that if a noncomplete graph G has a Hamiltonian path, then $pc(G) = 2$ in [1]. Hence, we can say that if the graph G is not a complete graph with $\delta(G) \geq \frac{n-1}{2}$, then $pc(G) = 2$. These results naturally lead the following two problems.

Problem 1.1 *Is there a constant $\frac{1}{2} \leq \alpha < 1$, such that if $d(x) + d(y) \geq \alpha n$ for every pair of nonadjacent vertices x, y of a graph G on n vertices, then $pc(G) = 2$?*

Problem 1.2 *Is there a constant $\frac{1}{4} < \beta < 1$, such that if $d(x) + d(y) \geq \beta n$ for every pair of nonadjacent vertices x, y of a bipartite graph G on n vertices, then $pc(G) = 2$?*

This kind of conditions is usually called the *degree sum condition*. Our main result in this paper is devoted to studying degree sum condition for a general graph or a bipartite graph to have proper connection number 2. As a result, the following conclusions are obtained.

Theorem 1.3 *Let G be a connected noncomplete graph of order $n \geq 5$ with $G \notin \{G_1, G_2, G_3\}$ shown in Figure 1. If $d(x) + d(y) \geq \frac{n}{2}$ for every pair of nonadjacent vertices $x, y \in V(G)$, then $pc(G) = 2$.*

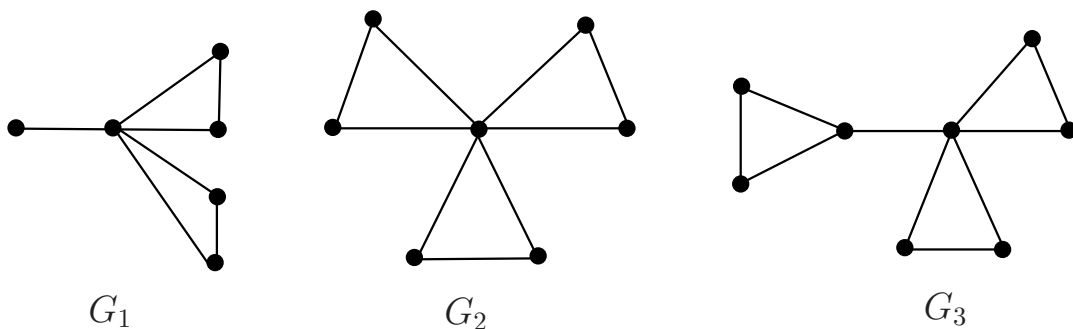


Figure 1: Three graphs of Theorem 1.3

Theorem 1.4 *Let G be a connected bipartite graph of order $n \geq 4$. If $d(x) + d(y) \geq \frac{n+6}{4}$ for every pair of nonadjacent vertices $x, y \in V(G)$, then $pc(G) = 2$.*

The following example shows that the degree sum condition in Theorems 1.3 are best possible. Let G_1 be a complete graph on 3 vertices and G_i be a complete graph with $\frac{n-3}{2}$ vertices ($n \geq 5$) for $i = 2, 3$. Then, take a vertex $v_i \in G_i$ for each $1 \leq i \leq 3$. Let G be a graph obtained from $G_1 \cup G_2 \cup G_3$ by joining v_1 and v_i with an edge for $i = 2, 3$. It is easy to see that $d(x) + d(y) \geq \frac{n}{2} - 1$ for every pair of nonadjacent vertices $x, y \in V(G)$, but $pc(G) = 3$.

To show that the degree condition in Theorem 1.4 is best possible, we construct the following example. Let G_i be a complete bipartite graph such that each part has 3 vertices and v_0 a vertex not in G_i for $i = 1, 2, 3$. Then, take a vertex $v_i \in G_i$ for each $1 \leq i \leq 3$. Let G be a bipartite graph obtained from $G_1 \cup G_2 \cup G_3$ by joining v_0 and v_i with an edge for each $1 \leq i \leq 3$. It is easily checked that $d(x) + d(y) \geq \frac{n}{4} + 1$ for every pair of nonadjacent vertices $x, y \in V(G)$, but $pc(G) = 3$.

2 Preliminaries

At the beginning of this section, we present some basic concepts as follows.

Definition 2.1 *Given a colored path $P = v_1v_2 \dots v_{t-1}v_t$ between any two vertices v_1 and v_t , we define $start(P)$ as the color of the first edge v_1v_2 in the path, and define $end(P)$ as the color of the last edge $v_{t-1}v_t$. In particular, if P is just the edge v_1v_t , then $start(P) = end(P) = c(v_1v_t)$.*

Definition 2.2 *Let c be a proper connection coloring of G . We say that G has the strong property under c if for any pair of vertices $u, v \in V(G)$, there exist two proper paths P_1, P_2 from u to v (not necessarily disjoint) such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$.*

Definition 2.3 *Let $B \subseteq E$ be the set of cut-edges of a graph G . We denote by \mathcal{D} the set of connected components of $G' = (V, E \setminus B)$. There exist two types of elements in \mathcal{D} , singletons and connected bridgeless subgraphs of G . we construct a new graph G^* that is obtained from contracting each element of \mathcal{D} of G' to a vertex. It is well-known that G^* is called the bridge-block tree of G . For the sake of simplicity, we call every element of \mathcal{D} a block of G . In particular, an element of \mathcal{D} which corresponds to a leaf in G^* is called a leaf-block of G .*

Definition 2.4 *A Hamiltonian path in a graph G is a path containing every vertex of G . And a graph having a Hamiltonian path is called a traceable graph.*

Definition 2.5 Let G be a graph with vertex set V . A vertex partition $V = V_1 \cup V_2 \cup \dots \cup V_k$ is called equitable if any two parts differ in size by at most one.

Next, we state some fundamental results on proper connection coloring which are used in the sequel.

Lemma 2.6 [1] If G is a nontrivial connected graph and H is a connected spanning subgraph of G , then $pc(G) \leq pc(H)$. In particular, $pc(G) \leq pc(T)$ for every spanning tree T of G .

Lemma 2.7 [1] If G be a traceble graph that is not complete, then $pc(G) = 2$.

Lemma 2.8 [1] Let G be a connected graph and v a vertex not in G . If $pc(G) = 2$, then $pc(G \cup v) = 2$ as long as $d(v) \geq 2$, that is, there are at least two edges connecting v to G .

Lemma 2.9 [3] If G is a bipartite connected bridgeless graph, then $pc(G) = 2$. Furthermore, there exists a 2-edge-coloring c of G such that G has the strong property under c .

The following result is an immediate consequence of Lemma 2.9.

Corollary 2.10 Let G be a connected bipartite graph and G^* be the bridge-block tree of G . If $\Delta(G^*) \leq 2$, then $pc(G) = 2$.

Proof. The proof proceeds by induction on the number of bridges of G . If G has no bridge, the result trivially holds by Lemma 2.9. Suppose that the result holds for every connected bipartite graph H with $r \geq 0$ bridges and $\Delta(H^*) \leq 2$. Let G be a connected bipartite graph with $r + 1$ bridges and $\Delta(G^*) \leq 2$. Note that G^* is a path. Thus, assume that $G^* = B_1 b_1 B_2 b_2 \dots B_r b_r B_{r+1} b_{r+1} B_{r+2}$ is a path, where b_i is a bridge of G for $i = 1, 2, \dots, r + 1$ and B_i is a block of G for $i = 1, 2, \dots, r + 2$. Let H be the subgraph of G such that $H^* = B_1 b_1 B_2 b_2 \dots B_r b_r B_{r+1}$. By the induction hypothesis, there exists a proper connection 2-coloring c of H with colors $\{1, 2\}$. In order to form a proper connection 2-coloring of G with colors $\{1, 2\}$, we only need to color the edges in $E(G) \setminus E(H)$. Without loss of generality, assume that $c(b_r) = 1$ under c . Let $b_r = uv$ and $b_{r+1} = wz$ with $v, w \in B_{r+1}$. If $v = w$, then color the edge b_{r+1} with color 2. If $v \neq w$, then there exists a proper path P between v and w in B_{r+1} under c , such that $start(P) = 2$. Next, we color the edge b_{r+1} satisfying $c(b_{r+1}) \neq end(P)$. At last, if B_{r+2} is not a singleton, applying Lemma 2.9 to B_{r+2} ,

there exists a proper connection 2-coloring c' of B_{r+2} with colors $\{1,2\}$ such that B_{r+2} has the strong property under c' . \square

Theorem 2.11 [6] *Let G be a connected noncomplete graph of order $n \geq 5$. If $G \notin \{G_2, G_3\}$ shown in Figure 1, and $\delta(G) \geq n/4$, then $pc(G) = 2$.*

Theorem 2.12 [6] *Let G be a connected bipartite graph of order $n \geq 4$. If $\delta(G) \geq \frac{n+6}{8}$, then $pc(G) = 2$.*

3 Proof of Theorem 1.3

Proof. The result trivially holds for $\delta(G) \geq \frac{n}{4}$ by Theorem 2.11. Next, we only need to consider $\delta(G) < \frac{n}{4}$ in the following. Let $X = \{x \mid d(x) = \delta(G)\}$. If $n = 5$, then $\delta(G) = 1$. Since $d(x) + d(y) \geq 3$ for every pair of nonadjacent vertices $x, y \in V(G)$, it follows that G has a Hamiltonian path. Thus, $pc(G) = 2$ by Lemma 2.7. If $n = 6$, then $\delta(G) = 1$. Take a vertex x_0 with $d(x_0) = 1$. Let $N(x_0) = \{y_0\}$, and $Y = V(G) \setminus \{x_0, y_0\} = \{y_1, y_2, y_3, y_4\}$. Since $d(x_0) + d(y_i) \geq 3$ for $i = 1, \dots, 4$, we have that $d(y_i) \geq 2$ for $i = 1, \dots, 4$. If there exists some y_i with $d(y_i) \geq 3$ for $1 \leq i \leq 4$, then it is easy to check that $pc(G) = 2$. If $d(y_i) = 2$ for $i = 1, \dots, 4$, then $G = G_1$ in Figure 1, a contradiction. Hence, it is sufficient to prove that the result holds for $\delta(G) < \frac{n}{4}$ and $n \geq 7$. Note that if G contains a bridgeless bipartite spanning subgraph H_0 , then $pc(G) \leq pc(H_0) \leq 2$ by Lemmas 2.6 and 2.9. Hence, assume that every bipartite spanning subgraph of G has a bridge. Let H be a bipartite spanning subgraph of G such that H has the maximum number of edges, and $\Delta(H^*)$ is as small as possible in the second place. If $\Delta(H^*) \leq 2$, $pc(G) \leq pc(H) = 2$ by Corollary 2.10. Next, assume that $\Delta(H^*) \geq 3$. To prove our result, we present the following fact.

Fact 1. Let $e = u_1u_2$ be a cut-edge of H , and let I_1 and I_2 be two components of $H - e$. Then $|E_G(I_1, I_2)| \leq 2$.

Suppose this is not true. Let (U_i, V_i) be the bipartition of I_i for $i = 1, 2$, such that $u_1 \in U_1$ and $u_2 \in U_2$. Noticing that $n \geq 7$, it is possible that there exists only one part U_i with $U_i = \{u_i\}$ and the corresponding $V_i = \emptyset$. Assume that there exists an edge $e_1 \in (E_G(U_1, U_2) \cup E_G(V_1, V_2)) \setminus e$, or there exist two edges $e_2, e_3 \in E_G(U_1, V_2) \cup E_G(V_1, U_2)$. Let $H_1 = H + e_1$ or $H_2 = H - e + e_2 + e_3$. It follows that H_i has $|E(H)| + 1$ edges for $i = 1, 2$, which contradicts the choice of H . Hence, $|E_G(I_1, I_2)| \leq 2$.

Let L be a leaf-block of H and b_L be the unique bridge incident with L in H . Applying Fact 1 to the cut-edge b_L , it follows that $|E_G(L, G \setminus V(L))| \leq 2$. Since

$b_L \in E_G(L, G \setminus V(L))$, it is obtained that $|E_G(L, L')| \leq 1$ for each pair of leaf-blocks L, L' of H . In order to complete our proof, we consider the following two cases.

Case 1. If $V(L) \cap X = \emptyset$ for any leaf-block L of H , then there exists a non-leaf block B_δ of H such that $x_0 \in B_\delta \cap X$. In this case, we claim that every leaf-block of H is not a singleton. Suppose it is not true. Let L_0 be a leaf-block of H with $V(L_0) = \{v_0\}$. It follows from Fact 1 that $d(v_0) \leq 2$, and $v_0 \notin X$. On the other hand, it is known that $d(v) \geq 2$ for each vertex v of each non-leaf block of H , which is impossible. Since every leaf-block of H is a maximal connected bridgeless bipartite subgraph, every leaf-block of H has at least 4 vertices. Let L be a leaf-block of H . Note that $|E_G(L, G \setminus V(L))| \leq 2$. Then, take a vertex v_L of L that is not adjacent to x_0 and $N(v_L) \subseteq V(L)$. Thus, $d(v_L) \geq \frac{n}{2} - d(x_0) = \frac{n}{2} - \delta(G)$, which implies that $|L| \geq \frac{n}{2} - \delta(G) + 1$. It follows that there exist at most three leaf-blocks of H . Otherwise, $|G| \geq 4 \times (\frac{n}{2} - \delta(G) + 1) + 1 > n + 5$, a contradiction. Hence, $\Delta(H^*) = 3$, and there is only one vertex $z_0 \in V(H^*)$ with $d_{H^*}(z_0) = 3$. We define B_0 as the block of H corresponding to z_0 .

Subcase 1.1. If B_0 is a singleton, we let $V(B_0) = \{b_0\}$ and e be a bridge incident to b_0 . Suppose that I_1 and I_2 are two components of $H - e$. Without loss of generality, assume that $b_0 \in I_2$, and let L be the leaf-block in I_1 . Bear in mind that $|E_G(I_1, I_2)| \leq 2$. If $d_{I_1}(b_0) = 1$, in this case we call e , the bridge incident to b_0 , a bridge of type *I*. Then $|I_1| \geq |L| \geq \frac{n}{2} - \delta(G) + 1$. If $d_{I_1}(b_0) = 2$, in this case we call e , the bridge incident to b_0 , a bridge of type *II*. Suppose that I_1 contains at least two blocks. Then $|I_1| \geq |L| + 1 \geq \frac{n}{2} - \delta(G) + 2$. Suppose that I_1 contains only L . Then there exist two vertices v_1, v_2 of L such that b_0 is adjacent to both v_1 and v_2 .

Claim 1. $|L| \geq \frac{n}{2} - \delta(G) + 2$ for L defined as above.

Suppose it is not true. Assume that $|L| \leq \frac{n}{2} - \delta(G) + 1$. Let $V(L) = \{v_1, v_2, u_1, \dots, u_t\}$ with $2 \leq t \leq \frac{n}{2} - \delta(G) - 1$. Since $|E_G(L, G \setminus V(L))| \leq 2$, and $v_i b_0 \in E_G(L, G \setminus V(L))$ for $i = 1, 2$, it follows that u_i is not adjacent to x_0 and $N(u_i) \subseteq V(L)$ for $i = 1, 2$, which implies that $d_L(u_i) \geq \frac{n}{2} - d(x_0) = \frac{n}{2} - \delta(G)$. Thus, $|L| = \frac{n}{2} - \delta(G) + 1$, and u_i is adjacent to all other vertices of L for $i = 1, \dots, t$. We can construct a new bipartite spanning subgraph H' of G by adding b_0 into L , such that b_0 and v_i lie in distinct equitable parts and are adjacent in the new block for $i = 1, 2$, which contradicts the maximality of H .

It follows that $|I_1| \geq |L| \geq \frac{n}{2} - \delta(G) + 2$. Let k be the number of bridges incident to b_0 of type *II*. Then $\delta(G) \leq d(b_0) \leq k + 3$. As a result, $|G| \geq 1 + 3 \times (\frac{n}{2} - \delta(G) + 1) + k > n + 1$, a contradiction.

Subcase 1.2. If B_0 is not a singleton, since B_0 is a maximal connected bridgeless

bipartite subgraph, B_0 has at least 4 vertices. Noticing that $d_{H^*}(z_0) = 3$, it is obtained that $|E_G(B_0, G \setminus V(B_0))| \leq 6$ by Fact 1. Then there exists at least one vertex b in B_0 satisfying that all but at most one of the neighbours of b are contained in B_0 . Hence, $|B_0| \geq d(b) \geq \delta(G)$. Consequently, $|G| \geq |B_0| + 3 \times (\frac{n}{2} - \delta(G) + 1) \geq \frac{3}{2}n - 2\delta(G) + 3 > n + 3$, a contradiction.

Case 2. There exists a leaf-block L_0 of H such that $x_0 \in L_0 \cap X$.

Claim 2. Let L' be a leaf-block of H with $E_G(L_0, L') = \emptyset$. Then, L' is not a singleton.

Suppose this is not true. Let L' be a leaf-block of H , such that $E_G(L_0, L') = \emptyset$ and $V(L') = \{v'\}$. It follows from Fact 1 that $d(v') \leq 2$. On the other hand, since x_0 is not adjacent to v' , we have that $d(v') \geq \frac{n}{2} - d(x_0) = \frac{n}{2} - \delta(G)$. Note that $n \geq 7$ and $\delta(G) < \frac{n}{4}$. If $n = 7$, then $\delta(G) = 1$. Hence, $d(v') \geq \frac{n}{2} - 1 = \frac{5}{2}$, a contradiction. If $n \geq 8$, then $d(v') \geq \frac{n}{2} - d(x_0) = \frac{n}{2} - \delta(G) > \frac{n}{4} \geq 2$, that is $d(v') \geq 3$, a contradiction.

Note that there exist at most two other leaf-blocks L_1, L_2 of H . Otherwise, there exist three other leaf-blocks L_1, L_2, L_3 of H . Since $|E_G(L_0, G \setminus V(L_0))| \leq 2$, there exist at least two leaf-blocks of L_1, L_2, L_3 , say L_1, L_2 , such that $E_G(L_0, L_i) = \emptyset$ for $i = 1, 2$. By Claim 2, L_i is not a singleton for $i = 1, 2$. Take a vertex v_i of L_i such that v_i is not adjacent to x_0 and $N(v_i) \subseteq V(L_i)$ for $i = 1, 2$. Thus, $|L_i| \geq \frac{n}{2} - \delta(G) + 1$ for $i = 1, 2$. It is easy to see that any leaf-block other than L_i, L_j has at least $\delta(G) - 1$ vertices. As a result, $|G| \geq 2 \times (\frac{n}{2} - \delta(G) + 1) + 2(\delta(G) - 1) + 1 = n + 1$, a contradiction. Hence, $\Delta(H^*) = 3$, and there is only one vertex $z_0 \in V(H^*)$ with $d_{H^*}(z_0) = 3$. We define B_0 as the block of H corresponding to z_0 . Let C_0, C_1, C_2 be the connected components of $H - V(B_0)$ such that L_i is the leaf-block contained in C_i for $i = 0, 1, 2$. Suppose that there exist two leaf-blocks L_i, L_j of H such that $E_G(L_i, L_j) = \{f\}$ for $0 \leq i \neq j \leq 2$. Let e_i be the unique bridge incident with both B_0 and C_i in H . Let $H_1 = H - e_i + f$. Note that H_1 is also a maximum bipartite spanning subgraph of G , but $\Delta(H_1^*) = 2$, which contradicts the choice of H . Thus, it is obtained that $E_G(L_i, L_j) = \emptyset$ for any two leaf-block L_i, L_j for $0 \leq i \neq j \leq 2$.

Subcase 2.1. If $\delta(G) \leq 2$, then $d(x_0) = \delta(G) \leq 2$. Since $E_G(L_0, L_i) = \emptyset$ for $1 \leq i \leq 2$, with the help of Claim 2, L_i is not a singleton for $i = 1, 2$. Consequently, $|L_i| \geq \frac{n}{2} - \delta(G) + 1$ for $i = 1, 2$. It follows that $|G| \geq |\{x_0 \cup N(x_0)\}| + |L_1| + |L_2| = (1 + \delta(G)) + 2 \times (\frac{n}{2} - \delta(G) + 1) = n + 3 - \delta(G) \geq n + 1$, a contradiction.

Subcase 2.2. If $\delta(G) \geq 3$, then $d(x_0) \geq 3$. First, we obtain that L_0 is not a singleton, which implies that $|L_0| \geq \delta(G) + 1$. Next, since $E_G(L_0, L_i) = \emptyset$ for $1 \leq i \leq 2$, it follows from Claim 2 that L_i is not a singleton for $i = 1, 2$, and so $|L_i| \geq \frac{n}{2} - \delta(G) + 1$ for $i = 1, 2$. Consequently, every leaf-block of H is not a singleton. With the similar argument in Case 1, we distinguish two cases based on

the condition that B_0 is or not a singleton. The unique different point is that there exists one leaf-block L_0 of H with $|L_0| \geq \delta(G) + 1$ in this case, and $|L| \geq \frac{n}{2} - \delta(G) + 1$ for each leaf-block L of H in Case 1. But the unique different point has no influence on proving our result. If B_0 is a singleton, it is worth mentioning that if the leaf-block L in Claim 1 is exactly L_0 , then the corresponding result is changed to $|L_0| \geq \delta(G) + 2$ in parallel. Using the similar argument in Subcase 1.1, we can obtain that $|G| \geq 1 + 2 \times (\frac{n}{2} - \delta(G) + 1) + (\delta(G) + 1) + k \geq n + 1$, where $k \geq \delta(G) - 3$, a contradiction. If B_0 is not a singleton, using the similar argument in Subcase 1.2, we can deduce that $|G| \geq |B_0| + 2 \times (\frac{n}{2} - \delta(G) + 1) + (\delta(G) + 1) \geq n + 3$, where $|B_0| \geq \delta(G)$, a contradiction. □

4 Proof of Theorem 1.4

Proof. The result trivially holds for $\delta(G) \geq \frac{n+6}{8}$ by Theorem 2.12. Next, we only need to consider $n \geq 4$ and $\delta(G) < \frac{n+6}{8}$ in the following. Let G^* be the bridge-block tree of G . If $\Delta(G^*) \leq 2$, then $pc(G) \leq 2$ by Corollary 2.10. Next, assume that $\Delta(G^*) \geq 3$ and let $X = \{x \mid d(x) = \delta(G)\}$.

Case 1. If $V(L) \cap X = \emptyset$ for any leaf-block L of G , then there exists a non-leaf block B_δ of G such that $x_0 \in B_\delta \cap X$. In this case, we claim that every leaf-block of G is not a singleton. Suppose it is not the case. Let L_0 be a leaf-block of G with $V(L_0) = \{v_0\}$. It follows that $d(v_0) = 1$, but $v_0 \notin X$, which is a contradiction. Since every leaf-block of G is a maximal connected bridgeless bipartite subgraph, every leaf-block of G has at least 4 vertices. Let L be a leaf-block of G . Note that $|E_G(L, G \setminus V(L))| = 1$. Then, take a vertex v_L of L that is not adjacent to x_0 and $N(v_L) \subseteq V(L)$ (there are at least $|L| - 1$ such vertices). Thus, $d(v_L) \geq \frac{n+6}{4} - d(x_0) = \frac{n+6}{4} - \delta(G)$. Since each part of L contains at least two vertices, which implies that $|L| \geq \frac{n+6}{2} - 2\delta(G)$. It follows that there exist at most three leaf-blocks of G . Otherwise, $|G| \geq 4 \times (\frac{n+6}{2} - 2\delta(G)) + 1 > n + 7$, a contradiction. Hence, $\Delta(G^*) = 3$, and there is only one vertex $z_0 \in V(G^*)$ with $d_{G^*}(z_0) = 3$. We define B_0 as the block of G corresponding to z_0 . If B_0 is a singleton, then $\delta(G) \leq d(b_0) = 3$. If B_0 is not a singleton, since B_0 is a maximal connected bridgeless bipartite subgraph, B_0 has at least 4 vertices. Consider the subgraph B_0 . We have that $\delta(B_0) \geq \delta(G) - 3$. Noticing that $d_{G^*}(z_0) = 3$, then there exists at least one vertex b in B_0 satisfying that all the neighbors of b are contained in B_0 . Hence, $|B_0| \geq \delta(G) + (\delta(G) - 3) = 2\delta(G) - 3$. Thus, no matter whether B_0 is or not a singleton, it always holds that $|B_0| \geq 2\delta(G) - 5$.

As a result, $|G| \geq 3 \times (\frac{n+6}{2} - 2\delta(G)) + |B_0| > n + 1$, a contradiction.

Case 2. There exists a leaf-block L_0 of G such that $x_0 \in L_0 \cap X$. If L_0 is a singleton, then $\delta(G) = d(x_0) = 1$. Let L be another leaf-block of G . Then L can not be a singleton. Suppose to the contrary, assume that $V(L) = \{v_0\}$, that is $d(v_0) = 1$. Noticing that x_0 is not adjacent to v_0 , then $d(x_0) + d(v_0) \geq \frac{n+6}{4} \geq \frac{5}{2}$, which is impossible. Hence, L has at least 4 vertices. It follows from $\Delta(G^*) \geq 3$ from that there exist at least two other leaf-blocks L_1, L_2 of G . Since $E_G(L_0, L_i) = \emptyset$, $d(v_i) \geq \frac{n+6}{4} - d(x_0) = \frac{n+6}{4} - 1$ for each vertex v_i of L_i for $i = 1, 2$, this means that $|L_i| \geq \frac{n+6}{2} - 2$. Thus, $|G| \geq |\{x_0 \cup N(x_0)\}| + |L_1| + |L_2| \geq 1 + 1 + 2 \times (\frac{n+6}{2} - 2) = n + 4$, a contradiction. If L_0 is not a singleton, since L_0 is a maximal connected bridgeless bipartite subgraph, L_0 has at least 4 vertices. In this case, it is easy to check that $|L_0| \geq 2\delta(G)$. Let L be another leaf-block of G . We claim that L is not a singleton. Suppose to the contrary, assume that $V(L) = \{v_0\}$, which implies $d(v_0) = 1$. Since $E_G(L_0, L) = \emptyset$, $d(v_0) \geq \frac{n+6}{4} - d(x_0) = \frac{n+6}{4} - \delta(G) > \frac{n+6}{8} \geq \frac{5}{4}$, a contradiction. Hence L has at least 4 vertices. Furthermore, $d(v) \geq \frac{n+6}{4} - d(x_0) = \frac{n+6}{4} - \delta(G)$ for each vertex v of L , which implies that $|L| \geq \frac{n+6}{2} - 2\delta(G)$. It follows that there exist at most two leaf-blocks of G other than L_0 . Otherwise, $|G| \geq 1 + 2\delta(G) + 3 \times (\frac{n+6}{2} - 2\delta(G)) > n + 7$, a contradiction. Hence, $\Delta(G^*) = 3$, and there is only one vertex $z_0 \in V(G^*)$ with $d_{G^*}(z_0) = 3$. We define B_0 as the block of G corresponding to z_0 . With similar argument in Case 1, we consider two cases based on the condition whether B_0 is or not a singleton. One can find that no matter what case occurs, it always holds that $|B_0| \geq 2\delta(G) - 5$. As a result, $|G| \geq 2 \times (\frac{n+6}{2} - 2\delta(G)) + 2\delta(G) + |B_0| \geq n + 1$, which is impossible. \square

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